JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 60 (1995) 3–12

Lower bounds for the R-order of convergence of simultaneous inclusion methods for polynomial roots and related iteration methods

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Received 10 September 1993, revised 1 February 1994

Abstract

The purpose of this paper is to give a survey on recent results concerning the estimation of the R-order of convergence of some classes of single-step iteration methods. We are dealing with some very general families of simultaneous inclusion methods for polynomial zeros and with error-recursions generalized by those of these families. It is shown how in these cases explicit lower bounds can be derived for the R-order depending on the parameters of the methods.

Keywords: Order of convergence; Polynomial roots

1. Introduction

The order of convergence can be used to characterize the local speed of convergence of an iterative process. By means of such an order of convergence it is possible to define an efficiency measure for iterative processes in order to make comparisons among them possible (see [18]). We are dealing here with the most common definition of an order of convergence, namely the so-called R-order of convergence as defined in [15].

We are considering iterative processes generating sequences $\{x_1^{(n)}\}_{n=0}^{\infty}, \dots, \{x_s^{(n)}\}_{n=0}^{\infty}$ of quantities, mostly real numbers or vectors, which approximate the unknown quantities $x_1^*, \dots, x_s^*$ in parallel or sequentially. For $s > 1$ we include simultaneous methods, which will be described more in detail later on, or parallel or multi-step methods if $x_1^* = x_2^* = \cdots = x_{h_1}^*$, $x_{h_1+1}^* = \cdots = x_{h_2}^*, \dots$ holds true. Denote by $\varepsilon_{n,i} = \|x_i^{(n)} - x_i^*\|$, $1 \leqslant i \leqslant s$ the norm of the errors,

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SSDI 0377-0427(94)00080-8

then we are assuming that a system of first-order difference inequalities in the following form

$$0 \leqslant \varepsilon_{n+1,i} \leqslant \sum_{k=1}^{k_i} \alpha_{ik} \prod_{j=1}^{s} \varepsilon_{n,j}^{m_{ijk}} \cdot \varepsilon_{n+1,j}^{r_{ijk}}, \quad 1 \leqslant i \leqslant s \tag{1.1}$$

can be derived for the iterative method in question. Here the exponents $m_{ijk} \geqslant 0$ and $r_{ijk} \geqslant 0$ are known as constants and $\alpha_{ik} > 0$. Many of the most common classes of iteration methods (like Newton-type methods) allow such local error estimations but not all iterative methods do have this property. In single-step type methods the condition $r_{ijk} = 0$ for $j \geqslant i$ and $1 \leqslant k \leqslant k_i$, $1 \leqslant i \leqslant s$ is fulfilled.

The question of how to compute the (best) R-order of convergence of all sequences $\{x_i^{(n)}\}_{n=1}^{\infty}$, $1 \leqslant i \leqslant s$ in terms of the coefficients m_{ijk} and r_{ijk} is dealt with in a series of papers by W. Burmeister and J.W. Schmidt (see, for example, [7] or [8]). For this purpose they are considering the following systems of linear inequalities:

$$\sum_{j=1}^{s} (m_{ijk} + \tau r_{ijk}) \cdot u_j \geqslant \tau u_i, \quad 1 \leqslant k \leqslant k_i, \quad 1 \leqslant i \leqslant s, \tag{1.2}$$

where

$$m_{ijk} \geqslant 0,\ r_{ijk} \geqslant 0, \quad 1 \leqslant k \leqslant k_i, \quad 1 \leqslant i,j \leqslant s.$$

If there exists a number $\tau > 1$ and a corresponding vector $u = (u_1, u_2, \ldots, u_s)$ with $u_i > 0$, $1 \leqslant i \leqslant s$ which both together fulfill the inequalities (1.2), then each sequence $\{x_i^{(n)}\}_{n=0}^{\infty}$, $1 \leqslant i \leqslant s$ converges at least of R-order τ provided some conditions on m_{ijk} and r_{ijk} hold (which is always the case for single-step or total-step methods) (see [8, Theorem 1]). But it still remains the frequently nontrivial task of determining such a proper number $\tau > 1$ in concrete cases. Sometimes it is even possible to calculate the greatest $\tau > 1$ which then is the optimal R-order of the sequences $\{x_i^{(n)}\}_{n=0}^{\infty}$, $1 \leqslant i \leqslant s$.

In the next Section 2 we will discuss a general model for simultaneous inclusion methods for polynomial zeros for which we can show that explicit lower bounds $\tau > 1$ can be derived for the R-order of convergence in terms of the parameters of the methods.

2. Lower bounds for the R-order of some simultaneous inclusion methods

In this section we consider the problem of the iterative inclusion of the roots $\xi_1, \xi_2, \ldots, \xi_s$ of a polynomial

$$p(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_0 = \prod_{i=1}^{s} (x - \xi_i)$$

by simultaneously computed sequences of intervals $\{X_i^{(n)}\}_{n=0}^{\infty}$, $1 \leqslant i \leqslant s$.

A compilation of such methods can be found in the monograph of Petković [16]. In the sequel we restrict our considerations to only real coefficients $a_i \in \mathbb{R}$, $1 \leqslant i \leqslant s-1$ as well as real roots ξ_i, $1 \leqslant i \leqslant s$ in order to avoid complex arithmetic. These in practice restrictive assumptions are only made to simplify the following formulations. In addition to this, we consider the case

ξ_i, $1 \leqslant i \leqslant s$, are simple roots,

since the formulas become slightly more complicated for known multiplicities of roots. All these simplifications do not restrict the generality of our approach, they just make it more transparent.

In recent time one very frequently uses interval arithmetic for the construction of inclusion methods. As initial values for such methods one assumes given inclusion intervals $\xi_1 \in X_1^{(0)}$, $\xi_2 \in X_2^{(0)}, \ldots, \xi_s \in X_s^{(0)}$ for the roots which are iteratively and simultaneously improved with $\lim_{n\to\infty} X_i^{(n)} = \xi_i$, $1 \leqslant i \leqslant s$. Interval arithmetic uses the four basic operations for (real) intervals $+, -, *$ and $\div$ defined by

$$X \star Y = \{x \star y \mid x \in X, y \in Y\}, \quad \star \in \{+, -, *, \div\}, \tag{2.1}$$

where $0 \notin Y$ in case of $\div$ (ranged arithmetic). The bounds of the resulting interval of an operation can easily be expressed by the bounds of the arguments. For more details see the literature, for example, [13, 1] or [14]. Implementations of interval operations in compilers can be found in [12, 11]. In [2, 3] the authors give an unified approach for an important class of higher-order simultaneous methods in the literature. By this approach, basic properties for numerical application such as

(i) inclusion monotonicity,
(ii) inclusion property,
(iii) convergence,
(iv) estimation of the R-order of convergence,

can be derived in a very general setting.

Since we are primarily interested here in the R-order of convergence of the methods, we only give a short description of this mathematical model and refer to the literature for more details.

Let $\varphi : \mathbb{R}^{s+k} \to \mathbb{R}$ be a real-valued function for $k \geqslant 1$ with the property

$$\varphi(x_1, x_2, \ldots, x_{s-1}, x_s, \ldots, x_{s+k}) = \varphi(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s-1)}, x_s, \ldots, x_{s+k}) \tag{2.2}$$

for every permutation π of $\{1, 2, \ldots, s-1\}$. As far as the authors know, all the formulas do have this property despite the fact that it is not so essential for the following considerations. With the help of the function φ we can derive a set of s real-valued functions by setting

$$\varphi_i(x_1, \ldots, x_{s+k}) = \varphi(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_s, x_i, x_{s+1}, \ldots, x_{s+k}), \quad 1 \leqslant i \leqslant s. \tag{2.3}$$

The functions φ_i, $1 \leqslant i \leqslant s$, shall be related to the zeros of the polynomial (1.1) by virtue of the fixed-point relations

$$\xi_i = \varphi_i(\xi_1, \ldots, \xi_{i-1}, x, \xi_{i+1}, \ldots, \xi_s, p(x), p'(x), \ldots, p^{(k-1)}(x)), \quad 1 \leqslant i \leqslant s. \tag{2.4}$$

Now, we would like to show that there exist such functions φ_i with (2.1)–(2.3).

Example 2.1.

(a) $$\varphi_i(s_1, \dots, x_{s+1}) = x_i - \frac{p(x_i)}{\prod_{j=1, j\neq i}^{s}(x_i - x_j)}, \quad [20].$$

(b) $$\varphi_i(x_1, \dots, x_{s+2}) = x_i - \frac{1}{\dfrac{p'(x_i)}{p(x_i)} - \sum_{j=1, j\neq i}^{s} \dfrac{1}{(x_i - x_j)}}, \quad [10].$$

(c) The formulas of Wang and Zheng [19].

Since we are primarily interested in interval arithmetic iteration functions, we assume that for each such function φ_i, $0 \leqslant i \leqslant s$, there exists an interval extension

$$\tilde{\varphi}_i(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_s, p(x_i), p'(x_i), \dots, p^{(k-1)}(x_i)),$$
$$x_i \in X_i, \quad 0 \leqslant i \leqslant s, \ (X_1, \dots, X_s) \subseteq (Y_1, \dots, Y_s) \tag{2.5}$$

for some interval vector $(Y_1, Y_2, \dots, Y_s)$. An interval extension can be achieved by substituting all arguments in the expression of the formulas by intervals and performing all occurring operations in interval arithmetic. In practice, this does not very much restrict the choice of the function φ since all known functions for our problem are rational functions. For rational functions there always exists such an interval extension.

Furthermore, this interval extension $\tilde{\varphi}_i$, $0 \leqslant i \leqslant s$, shall have the property

$$d(\tilde{\varphi}_i(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_s, p(x_i), p'(x_i), \dots, p^{(k-1)}(x_i)))$$
$$\leqslant \kappa d(X_i)^\alpha \left(\sum_{j=1, j\neq i}^{s} d(X_j)^\beta \right), \quad \alpha, \beta \in \mathbb{N}\backslash\{0\}, \quad 1 \leqslant i \leqslant s, \tag{2.6}$$

where $d(X) = d([x_1, x_2]) = x_2 - x_1$ is the width of the interval X. Assumption (2.6) is used in the estimation of the order of convergence of our iteration methods.

Remark 2.2. The more general property

$$d(\tilde{\varphi}_i(X_1, \dots, X_{i-1}, x_i, X_{i+1}, \dots, X_s, p(x_i), p'(x_i), \dots, p^{(k-1)}(x_i)))$$
$$\leqslant \sigma d(X_i)^\alpha \left(\sum_{j=1, j\neq i}^{s} d(X_j) \right)^\beta, \quad \alpha, \beta \in \mathbb{N}\backslash\{0\}, \quad 1 \leqslant i \leqslant s, \tag{2.7}$$

is more difficult to be treated with respect to the speed of convergence due to the “mixed terms”. But in view of the inequality

$$(x_1 + x_2 + \cdots + x_s)^\beta \leqslant \delta(x_1^\beta + x_2^\beta + \cdots + x_s^\beta), \quad \beta \in \mathbb{N}\backslash\{0\}, \ x_i \geqslant 0, \ 1 \leqslant i \leqslant s, \tag{2.8}$$

we always get a lower bound for the speed of convergence by replacing (2.7) by (2.6).

An example showing how such an estimation (2.6) can easily be derived for special functions $\tilde{\varphi}_i, 0 \leqslant i \leqslant s$, by simple applications of the well-known rules for d in interval arithmetic can be found in [4].

We are now able to define fixed-point iteration methods (simultaneous methods) in the form of total-step and single-step methods:

(total-step method)

$$X_i^{(n+1)} = \tilde{\varphi}_i(X_1^{(n)}, \ldots, X_{i-1}^{(n)}, x_i^{(n)}, X_{i+1}^{(n)}, \ldots, X_s^{(n)}, p(x_i^{(n)}) \ldots, p^{(k-1)}(x_i^{(n)})) \cap X_i^{(n)},$$
$$x_i^{(n)} \in X_i^{(n)}, \quad 0 \leqslant i \leqslant s, \quad n \geqslant 0, \tag{2.9}$$

(single-step method)

$$X_i^{(n+1)} = \tilde{\varphi}_i(X_1^{(n+1)}, \ldots, X_{i-1}^{(n+1)}, x_i^{(n)}, X_{i+1}^{(n)}, \ldots, X_n^{(n)}, p(x_i^{(n)}), \ldots, p^{(k-1)}(x_i^{(n)})) \cap X_i^{(n)},$$
$$x_i^{(n)} \in X_i^{(n)}, \quad 0 \leqslant i \leqslant s, \quad n \geqslant 0. \tag{2.10}$$

For methods (2.9) and (2.10) we can prove the following theorem (see [2, 3]).

Theorem 2.3. *Iteration methods* (2.9) *and* (2.10) *have the properties*

(i) $X_i^{(0)} \supseteq X_i^{(1)} \supseteq X_i^{(2)} \supseteq \cdots$, $1 \leqslant i \leqslant s$;

(ii) $\xi_i \in X_i^{(n)}$, $1 \leqslant i \leqslant s \Rightarrow \xi_i \in X_i^{(n+1)}$, $1 \leqslant i \leqslant s$ *(inclusion property); this ensures in case* $\xi_i \in X_i^{(0)}$, $1 \leqslant i \leqslant s$, *that the intersection can never become empty and thus methods* (2.9) *and* (2.10) *are defined for* $n \geqslant 0$;

(iii) *methods* (2.9) *and* (2.10) *are locally convergent if* $\xi_i \in X_i^{(0)}$, $1 \leqslant i \leqslant s$, *i.e., for sufficiently small values* $d(X_i^{(0)})$, $1 \leqslant i \leqslant s$, *we have* $X_i^{(n)} \to \xi_i$, $1 \leqslant i \leqslant s$;

(iv) *the R-order of convergence of every sequence* $\{X_i^{(n)}\}$, $1 \leqslant i \leqslant s$ *(which is the R-order of convergence of* $\{d(X_i^{(n)})\}$, $1 \leqslant i \leqslant s$*) of method* (2.9) *is at least* $\alpha + \beta$ *and that of method* (2.10) *is at least* τ^* *where* $\tau^* = \varepsilon\beta + \alpha$ *and* ε *is the uniquely positive root of the polynomial equation* $\varepsilon^s - \beta\varepsilon - \alpha = 0$ *and* $\varepsilon > 1$.

Remark 2.4. From the definition of the polynomial of ε and from the following derivation of bounds for ε it follows that $\varepsilon \to 1$ as s tends to infinity. This means that $\tau^* \to \alpha + \beta$, the R-order of convergence of the corresponding total-step methods, as s tends to infinity. It can even be shown that the convergence of the τ^* as a function of s is monotonic. Let ε_s be the positive solution of the equation $\varepsilon^s - \beta \cdot \varepsilon - \alpha = 0$, then $\varepsilon_s^{s+1} - \beta \cdot \varepsilon_s - \alpha = \varepsilon_s \cdot (\varepsilon_s^s - \beta \cdot \varepsilon_s - \alpha) + \beta \cdot (\varepsilon_s^2 - \varepsilon_s) + \alpha \cdot (\varepsilon_s - 1) > 0$ since $\varepsilon_s > 1$ which means $\varepsilon_{s+1} < \varepsilon_s$.

Furthermore, the value of ε increases with increasing parameters α and β. This can be shown as follows: Let ε be the positive root of the equation $\varepsilon^s - \beta\varepsilon - \alpha = 0$. Then we choose $\alpha' = \alpha + a$ and $\beta' = \beta + b$ with $a > 0$ and $b > 0$. Since $\varepsilon^s - (\beta + b)\varepsilon - (\alpha + a) = \varepsilon^s - \beta\varepsilon - \alpha - (b\varepsilon + a) < 0$ it follows that for the positive root k of the equation $k^s - \beta' k - \alpha' = 0$ the relation $k > \varepsilon$ holds true.

Thus, with increasing parameters α and β the bound τ^* (given by $\beta\varepsilon + \alpha$) is increasing. Passing from total-step to single-step method one can improve the R-order of convergence by at least 1 if $\beta \geqslant 1/(\varepsilon - 1)$ which might be fulfilled for reasonable parameters especially if n is not too large. (This would surely be the case, for example, for a hypothetic method with $\alpha = 6$ and $\beta = 1$ for $s = 3$.)

Despite the fact that from the defining equation for τ^* (or ε) the values can easily be calculated for given parameters s, α and β by applying Newton's method, it is sometimes necessary to have explicit bounds for τ^* depending on s, α and β. This is the case when, for example, efficiency measures are involved.

We first give such explicit lower and upper bounds for the values of τ^*. For this purpose, we observe that the polynomial for defining τ^*,

$$q(\tau) = \left(\frac{\tau-\alpha}{\beta}\right)^s - \beta\left(\frac{\tau-\alpha}{\beta}\right) - \alpha, \tag{2.11}$$

is surely convex in the interval $[\alpha+\beta, +\infty)$ if $s > \beta$. In most of the formulas in the literature where the values of β are usually not so large this condition is fulfilled and, on the other hand, the degrees s of the polynomials dealt with are at least of moderate size. The convexity of $q(\tau)$ follows from

$$q'(\tau) = \frac{s}{\beta}\left(\frac{\tau-\alpha}{\beta}\right)^{s-1} - 1 > 0 \quad \text{for } \tau > \alpha+\beta \text{ if } s > \beta$$

and

$$q''(\tau) = \frac{s(s-1)}{\beta^2}\left(\frac{\tau-\alpha}{\beta}\right)^{s-2} > 0 \quad \text{for } \tau > \alpha+\beta \geqslant \alpha.$$

Since $q(\alpha+\beta) = 1-(\alpha+\beta) < 0$, we have $\tau^* > \alpha+\beta$. If we apply one step of Newton's method to the polynomial $q(\tau)$ at $\tau = \alpha+\beta$, then we get a new approximation $\tau' > \tau^*$. It is easily shown that $\tau' = \alpha+\beta+\beta(\alpha+\beta-1)/(s-\beta)$. Then it follows from the convexity of $q(\tau)$ that a secant-step performed with $\tau = \alpha+\beta$ and τ' gives a new lower bound $\tau'' < \tau^*$. A simple arithmetic leads to

$$\tau'' = \alpha+\beta+\frac{((\alpha+\beta)-1)(\beta\cdot((\alpha+\beta)-1)/(s-\beta))}{(1+((\alpha+\beta)-1)/(s-\beta))^s+\beta\cdot((\alpha+\beta)-1)/(s-\beta)-1} < \tau^*. \tag{2.12}$$

Again, these bounds for τ^* prove that $\tau^* \to \alpha+\beta$ as s tends to ∞. The lower bound (2.12) for τ^* mirrors all the properties of τ^*, but there is still the restriction $s > \beta$. This can be overcome if we choose a different way of estimating the root τ^*. In doing so, we remember from the proof of Theorem 2.3 (iv) in [3] that the polynomial (2.11) was derived as the characteristical polynomial of the nonnegative irreducible matrix

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & \cdots & 0 \\ 0 & \alpha & \beta & 0 & \cdots & 0 \\ & & \ddots & \ddots & 0 & \\ & 0 & & \ddots & \ddots & \\ & & & & \ddots & \beta \\ \alpha\cdot\beta & \beta^2 & 0 & \cdots & 0 & \alpha \end{pmatrix}.$$

τ^* was in [3] the spectral radius of A due to the theory of Burmeister and Schmidt and is, according to the Perron–Frobenius theorem, the positive root of the polynomial (2.10) which is unique by Descartes' rule of signs.

The following lemma of Deutsch [9] enables us to estimate the spectral radius of A and thus to give a lower bound for τ^*.

Lemma 2.5. *Let $A = (a_{ij})$ be a nonnegative irreducible $n \times n$ matrix and let x and y be positive vectors satisfying*

$$Ax = Dx \quad \text{and} \quad A^{\mathrm{T}}y = Dy$$

for some positive diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$. If x is not a Perron vector of A, then

$$\varrho(A) > \frac{y^{\mathrm{T}}Dx}{y^{\mathrm{T}}x}.$$

For the application of this lemma to the matrix A given above we have to choose a proper vector x. Let us choose $x = (1, \dots, 1)^{\mathrm{T}}$, then we get $d_i = \sum_{j=1}^{s} a_{ij}$, $1 \leqslant i \leqslant s$. Thus,

$$D = \operatorname{diag}(\alpha + \beta, \dots, \alpha + \beta, \beta(\alpha + \beta) + \alpha).$$

Solving $(A^{\mathrm{T}} - D)y = 0$ gives

$$y = \gamma(\alpha, \alpha + \beta, \dots, \alpha + \beta, 1)^{\mathrm{T}}.$$

Then we get $y^{\mathrm{T}}x = \gamma((s-2)(\alpha + \beta) + \alpha + 1)$ and $y^{\mathrm{T}}Dx = \gamma((\alpha + \beta)^2 + (s-2)(\alpha + \beta)^2 + \alpha)$ and finally

$$\tau^* > \frac{(\alpha + \beta) + ((\alpha + \beta)/(s-2)) + \alpha/((s-2)(\alpha + \beta))}{1 + (\alpha + 1)/((s-2)(\alpha + \beta))} > \alpha + \beta. \tag{2.13}$$

Lemma 2.5 of Deutsch was used for calculating lower bounds for the R-order of simultaneous methods in [17].

3. Lower bounds for the R-order of more general classes of iteration methods

In this section we go back to error recursion (1.1). For some special sets of parameters m_{ijk} and r_{ijk} it is possible to give a lower bound for the R-order. This means that we have to construct a pair of a value τ and a vector $u > 0$, both fulfilling the inequalities (1.2). The authors have proved in [5] the following lemma.

Lemma 3.1. *Let the coefficients of the inequalities* (1.2) *fulfill the following conditions*:

(i) $a \leqslant m_{ssk} + r_{ssk}$, $1 \leqslant k \leqslant k_i$,

(ii) $a < c \leqslant \sum_{j=1}^{s}(m_{ijk} + r_{ijk})$, $1 \leqslant k \leqslant k_i$, $1 \leqslant i \leqslant s-1$ *with $b = c - a$ and $a + b > 1$,*

and let $d_k = \sum_{j=1}^{s-1} r_{sjk}$ and $d = \min_{1 \leqslant k \leqslant k_s} d_k$ then there exists a number $\tau^* > a + b$ and a vector u with components

$$u_i = \left(\frac{\tau^* - a}{b}\right)^{i-1} > 0, \quad 1 \leqslant i \leqslant s,$$

which both together are feasible solutions of the inequalities (1.2). τ^* *is the unique positive root of the polynomial equation*

$$\left(\frac{\tau-a}{b}\right)^s-\left(\frac{\tau-a}{b}\right)\cdot d-\frac{a}{b}\cdot d=0. \tag{3.1}$$

Remark 3.2. As in Section 2, the lower bound τ^* can be written as $\tau^* = a + \sigma \cdot d$, where $\sigma > 1$ is the unique positive root of the polynomial $p(\sigma) = \sigma^s - d\cdot\sigma - a/b\cdot d$. In particular, this means that $\sigma^* > a + d$.

For given values of s, a, b and d the number τ^* can easily be calculated by an application of Newton's method. But if we want an explicit expression for a lower bound of the R-order in dependence of the parameters s, a, b and d, then we have to do similar conclusions as in Section 2.

In order to derive an explicit lower bound for τ^*, we try to apply Lemma 2.5. Since the structure of the polynomial in (3.1) is similar to that in (2.11), we easily verify that the nonnegative irreducible matrix

$$A=\begin{pmatrix} a & d & & & & \\ & a & b & & & 0 \\ & & a & \ddots & & \\ & 0 & & \ddots & \ddots & \\ & & & & \ddots & b \\ ab & db & 0 & \cdots & 0 & a \end{pmatrix}$$

has (3.1) as its characteristic equation. According to the Perron–Frobenius theorem, the spectral radius of A is a positive root of Eq. (3.1). By Descartes' rule of sign this equation has only one unique positive root τ^* which is the spectral radius of the matrix A. Now, applying Lemma 2.5 to the matrix A, we get a lower bound for τ^*.

Again we choose in Lemma 2.5, $x = (1, \ldots, 1)^{\mathrm{T}}$ and get

$$d_i=\sum_{j=1}^{s} a_{ij},\quad 1\leqslant i\leqslant s$$

or

$$D=\operatorname{diag}(a+d,\ a+b,\ldots,a+b,\ (a+d)b+a).$$

Using these quantities we solve the system $(A^{\mathrm{T}} - D)y = 0$ with

$$y=\gamma\left(\frac{ab}{d},\ a+d,\ldots,a+d\right)^{\mathrm{T}}.$$

and thus get the values

$$y^{\mathrm{T}}Dx = \gamma(a+d)\left(\frac{ab}{c} + (s-1)(a+b)\right),$$

$$y^{\mathrm{T}}x = \gamma\left(\frac{ab}{d} + (s-1)(a+d)\right).$$

Finally, we get the lower bound

$$\tau^* > \frac{(a+b)[(ab/d)+(s-1)(a+b)]}{(a-1)(a+d)+(ab/d)}. \tag{3.2}$$

4. Conclusions

In [5] the authors give a very general principle for the construction of classes of error recursions for which bounds the R-order can be given. The derivation of explic it lower bounds as described in the examples of Section 2 and Section 3 can also be applied in such cases. In particular, it is possible to make the same considerations for the so-called combined methods in [3] which have a lower bound for the R-order in terms of a spectral radius of a certain nonnegative matrix. It should be stressed here that our lower bounds mirror some basic properties of the value τ^* as there are: the asymptotic behavior for large degrees s and the upper bound property for certain values (which sometimes can be interpreted as the R-order of total-step methods). Therefore they can substitute the real values of τ^* for many purposes.

Acknowledgement

The authors would like to express their gratitude to one of the referees for detecting numerous typing errors in the original manuscript.

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